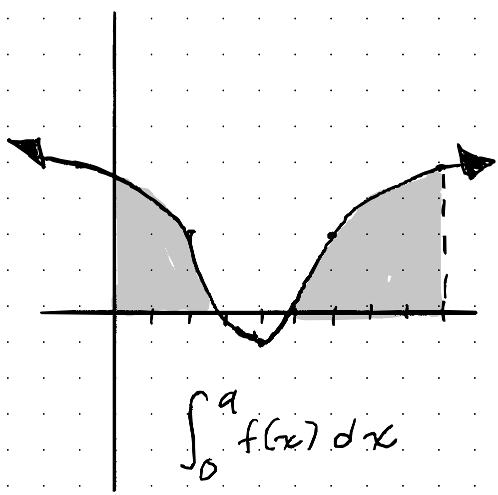
AP Calculus AB

2nd Semester

$$\frac{d}{dx}\left(f(x)\right)$$

$$\int_a^b f(x)\,dx$$



$$\int_0^a f(x)\,dx$$

4.5 Integration by Substitution

Antidifferentiation of Composite Functions:

Let "g" be a function, with a range of "I"; "f" is also a function which is continuous on "I". If "g" is differentiable on it's domain & "F" is an antiderivative of "f" on "I," then:

$$\int f(g(x))g'(x)\,dx = F(g(x)) + C$$

$$\int f(u)\,du = F(u) + C$$

Ex 1: $\int (x^2+1)^2(2x)\,dx$ Recognizing the $f(g(x))g'(x)$ pattern:

$\to u = x^2+1 \to du = 2x\,dx \to \int \underbrace{(x^2+1)}_{u}{}^{2}\underbrace{(2x)\,dx}_{du} \to \int u^2\,du \to$

$= \frac{u^3}{3} + C \to \frac{(x^2+1)^3}{3} + C$

Ex 2: $\int 5\cos 5x\,dx$ $u = 5x$ $du = 5\,dx$

$\to \int \cos u\,du \to -\sin u + C \to \boxed{-\sin 5x + C}$

Ex 3: $\int x(x^2+1)^2\,dx \to$ $u = x^2+1$ $du = 2x\,dx$

$\frac{1}{2}\int x(x^2+1)^2(2)\,dx \to \frac{1}{2}\int u^2\,du = \frac{1}{2}\left(\frac{u^3}{3}\right) + C \to \boxed{\frac{(x^2+1)^3}{6} + 3}$

Ex 4: $\int \sqrt{2x-1}\,dx \to$ $u = 2x-1$ $du = 2\,dx \to$

$\frac{1}{2}\int u^{1/2}\,du \to \frac{1}{2}\left(\frac{2u^{3/2}}{3}\right) + C \to \frac{u^{3/2}}{3} + C = \boxed{\frac{(2x+1)^{3/2}}{3} + C}$

Ex 5: $\int (3-x^4)^6(4x^3)\,dx \to u = 3-x^4$ $du = -4x^3\,dx$

$-1\int u^6\,du = -1\left(\frac{u^7}{7}\right) + C \to \frac{-u^7}{7} + C \to \boxed{\frac{-(3-x^4)^7}{7} + C}$

Warm up and review Jan 14 2025

1. $\int_1^9 \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx \rightarrow u = 1+\sqrt{x}$, $\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, $du = \frac{1}{2\sqrt{x}}dx$

$x^{1/2} \rightarrow \frac{1}{2\sqrt{x}}$

$2\int \frac{1}{u^2}du$

u-limits:

$x=1$: $u = 1+\sqrt{1} = 2$

$x=9$: $u = 1+\sqrt{9} \rightarrow 1+3 = 4$

$$2\int_2^4 u^{-2}\,du \rightarrow 2\left[\frac{u^{-1}}{-1}\right]_2^4 \rightarrow -2\left[\frac{1}{u}\right]_2^4$$

$$-2\left[\frac{1}{4}-\frac{1}{2}\right] \rightarrow -2\left[-\frac{1}{4}\right] \rightarrow \frac{1}{2}$$

2. $\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx \rightarrow u = 2x-1 \rightarrow 2$, $\frac{du}{dx} = 2 \rightarrow du = 2dx$

$\rightarrow x = \frac{1+u}{2} \rightarrow \frac{1}{2}\int_1^5 \frac{x}{\sqrt{2x-1}}\,dx(2)$

u-limits:

$x=1$: $2(1)-1 = 1$

$x=5$: $2(5)-1 = 9$

$$\rightarrow \frac{1}{2}\int_1^9 \frac{\left(\frac{u+1}{2}\right)}{u^{1/2}}\,du \rightarrow \frac{1}{4}\int_1^9 \left(\frac{u+1}{u^{1/2}}\right)du$$

$$= \frac{1}{4}\int_1^9 \left(u^{1/2}+u^{-1/2}\right)du \rightarrow \frac{1}{4}\left[\frac{2u^{3/2}}{3}+2\sqrt{u}\right]_1^9$$

$$\rightarrow \frac{1}{4}\left[\left(\frac{54}{3}+6\right)-\left(\frac{2}{3}+2\right)\right] \rightarrow \frac{1}{4}\left(\frac{72}{3}-\frac{8}{3}\right) = \frac{1}{4}\left(\frac{64}{3}\right)$$

$\left(\frac{2}{3}+2\right) \rightarrow \frac{8}{3}$

$$\rightarrow \frac{16}{3}$$

Review of Final from December

$\sin(-u) = -\sin(u)$

learn to find limits in functions using conjugates

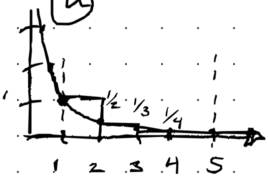
4.2 Areas

Jan 16

Warm up: $f(x)=\frac{1}{x}$, $[1,5]$

mid point: 1.575
right: 1.283

(a)



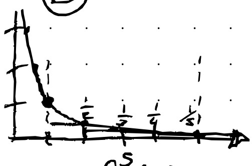
$(1\times1)+(1\times\frac{1}{2})+(1\times\frac{1}{3})+(1\times\frac{1}{4})$

$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$

$\to 1+\frac{3}{4}+\frac{1}{3} \to 1+\frac{13}{12}$

$\frac{9}{12}+\frac{4}{12}$ $\quad 2+\frac{1}{12} \to \frac{25}{12}$

(b)



$A=\int_1^5 \left(\frac{1}{x}\right) dx$

$\to = 1\left(f(2)+f(3)+f(4)+f(5)\right)$

$=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}$

$=\frac{77}{60}$ $\quad 1.283$

(a) 2.083

(b) 1.283

lower sum $<$ actual area $<$ upper sum
(inscribed) (circumscribed)

What is a sigma notation?

$$\text{sum of}\left\{\sum_{i=\ldots}^{n} \text{expression in terms of } i\right.$$

$n \to$ last # to plug in?

$i = \ldots \to$ initial # to plug in

★ The sum of n terms $a_1, a_2, a_3, \ldots\ldots, a_n$ is written as:

$$\sum_{i=1}^{n} a_i = a_1, a_2, a_3, \ldots a_n$$

i = index of summation

Ex 1:

(a) $\sum_{i=1}^{6} i = 1+2+3+4+5+6=21$

(b) $\sum_{i=0}^{5}(i+1) = (0+1)+(1+1)+(2+1)+(3+1)+(4+1)+(5+1) = 21$

(c) $\sum_{j=3}^{7} j^2 = 3^2 + 4^2 + 5^2 + 6^2 + 7^2 = 135$

(d) $\sum_{j=1}^{5} \frac{1}{\sqrt{j}} = \frac{1}{1} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{5}} = 3.232$

Properties of Summation

(1) $\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i$

(2) $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$

Summation Formulas

(1) $\sum_{i=1}^{n} c = cn$, c is a constant

(2) $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ (3) $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

(4) $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$

Ex 2: $\sum_{i=1}^{n} \frac{i+1}{n^2}$ for $n = 10, 100, 1000, 10.000$

$= \frac{1}{n^2}\sum_{i=1}^{n}(i+1) = \frac{1}{n^2}\left(\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$

$\rightarrow \frac{1}{n^2}\left(\frac{n(n+1)}{2} + n\right)$ → simplify $\rightarrow \frac{1}{n}\left(\frac{(n+1)}{2} + 1\right)\cdot\frac{2}{2}$

$\rightarrow \frac{1}{n}\left(\frac{n+3}{2}\right) \rightarrow \boxed{\frac{n+3}{2n}}$

sum

$n=10 \rightarrow \frac{10+3}{2(10)} \rightarrow \frac{13}{20}$

$n=100 \rightarrow \frac{100+3}{2(100)} \rightarrow \frac{103}{200}$

Upper Sums & Lower Sums

The sum of areas of the inscribed rectangles is called a lower sum, & the sum of the areas of the circumscribed rectangles is called an upper sum

$$\text{Lower sum} = s(n) = \sum_{i=1}^{n} f(m_i)\,\Delta x$$

$$\text{Upper sum} = S(n) = \sum_{i=1}^{n} f(M_i)\,\Delta x$$

$$m_i = a + (i-1)\Delta x \qquad M_i = a + i\Delta x$$

$$\Delta x = \frac{b-a}{n}$$

Ex 3: Find upper & lower sums for region bounded by $f(x) = x^2$ & the x-axis between $[0, 2]$

$$\text{Upper sum} = \sum_{i=1}^{n} f(M_i)\,\Delta x$$

$$\Delta x = \frac{2-0}{n} = \frac{2}{n}$$

$$M_i = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

$$\to \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2i}{n}\right)^2$$

$$\to \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4i^2}{n^2}\right) \to \frac{8}{n^3}\sum_{i=1}^{n}\left(i^2\right)$$

$$= \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6}\right) \to \frac{4(n+1)(2n+1)}{3n^2}$$

$$\to \frac{4(2n^2+3n+1)}{3n^2} \to \boxed{\frac{8n^2+12n+4}{3n^2}}$$

Upper Sum

Lower Sum $\underset{a}{[0},\underset{b}{2]}$ $f(x) = x^2$

$$\text{lower sum} = \sum_{i=1}^{n} f(m_i)\,\Delta x$$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} \rightarrow \frac{2}{n}$$

$$m_i = a + (i-1)\Delta x = 0 + (i-1)\frac{2}{n} = \frac{2(i-1)}{n}$$

$$= \sum_{i=1}^{n} f\left(\frac{2(i-1)}{n}\right)\frac{2}{n}$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{2(i-1)}{n}\right)^2 \qquad \left(\frac{2(i-1)}{n}\right)^2$$

$$\frac{2}{n}\cdot\frac{4}{n^2} \qquad \frac{4}{n^2}(i^2 - 2i^2 + 1)$$

$$= \frac{8}{n^3}\sum_{i=1}^{n}(i^2 - 2i^2 + 1) = \frac{8}{n^3}\left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1\right)$$

$$\rightarrow \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + n\right)$$

$$\rightarrow \frac{8}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\left(\frac{n(n+1)}{2}\right) + \frac{n}{1}\right)$$

$$\rightarrow \frac{8}{n^2}\left(\frac{(n+1)(2n+1)}{6} - 2\left(\frac{(n+1)}{2}\right) + \frac{1}{1}\right)$$

$$\rightarrow \frac{8}{n^2}\left(\frac{(n+1)(2n+1)}{6} - \frac{6(n+1)}{6} + \frac{6}{6}\right) \quad 3n$$

$$\rightarrow \frac{8}{n^2}\left(\frac{2n^2 + \overbrace{n + 2n}+1}{6} - \frac{6n+6}{6} + \frac{6}{6}\right)$$

$$\rightarrow \frac{8}{n^2}\left(\frac{2n^2 - 3n + 1}{6}\right) = \frac{16n^2 - 24n + 8}{6n^2} \rightarrow \boxed{\frac{8n^2 - 12n + 4}{3n^2}}$$

$$\lim_{n\to\infty}(S_n) = \frac{8}{3}$$

Warm up Jan 17

42 $\int_0^3 f(x)\,dx = 4$ & $\int_3^6 f(x)\,dx = -1$

a $\int_0^6 f(x)\,dx = 3$ ✓

b $\int_6^3 f(x)\,dx = 1$ ✓

c $\int_3^3 f(x)\,dx = 0$ ✓

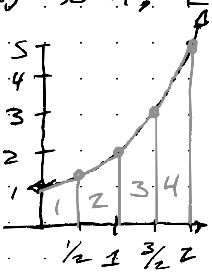
d $\int_3^6 -5f(x)\,dx = -5(-1) = 5$ ✓

4.6 Trapezoidal Rule

Learn the Trapezoidal Rule:

Find the area under the curve using 4 trapezoids

$f(x)=x^2+1;\ [0,2]$



$$A=\int_0^2 f(x^2+1)\,dx$$

$$\rightarrow \underbrace{\frac{1}{2}\left(\frac{1}{2}\right)\left(f(0)+f\left(\frac{1}{2}\right)\right)}_{\text{Trap. 1}} \rightarrow \frac{1}{4}\left(1+\left(\left(\frac{1}{2}\right)^2+1\right)\right)$$

$$A_{\text{trapezoid}}=\frac{1}{2}h(b_1+b_2)$$

$$\Delta x\left(f_{left}+f_{right}\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f\left(\frac{1}{2}\right)+f(1)\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f(1)+f\left(\frac{3}{2}\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{2}\right)\left(f\left(\frac{3}{2}\right)+f(2)\right)\rightarrow$$

$f(0)=1 \quad f\left(\frac{1}{2}\right)=\frac{5}{4}$

$f(1)=2 \quad f\left(\frac{3}{2}\right)=\frac{13}{4}$

$f(2)=5$

$$\frac{1}{4}\left[\left(f(0)+f\left(\frac{1}{2}\right)\right)+\left(f\left(\frac{1}{2}\right)+f(1)\right)+\left(f(1)+f\left(\frac{3}{2}\right)\right)+\left(f\left(\frac{3}{2}\right)+f(2)\right)\right]$$

$$\rightarrow \frac{1}{4}\left(f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+f(2)\right)$$

$$\rightarrow \frac{1}{4}\left(1+2\left(\frac{5}{4}\right)+2(2)+2\left(\frac{13}{4}\right)+5\right)$$

$$\rightarrow \frac{1}{4}(19) \rightarrow \frac{19}{4}$$

The Trapezoidal Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left[f(x_0)+2f(x_1)+2f(x_2)+\cdots+f(x_n)\right]$$

As "n" approaches ∞, leftside approaches $\int_a^b f(x)\,dx$

Always Remember:

1. You can only use the trapezoidal rule if $f(x)$ is given
2. All trapezoides must be the same width.

If these rules aren't met, use simple geometry:

$$A_{trap.} = \frac{1}{2}\Delta x(b_1 + b_2)$$

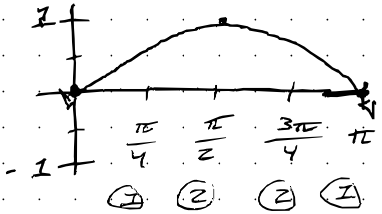
Ex 1:

use $n=4$

$$\int_0^{\pi} \sin x \, dx \approx \frac{\pi - 0}{8}\left[f(0) + 2f\left(\tfrac{\pi}{2}\right) + 2f\left(\tfrac{3\pi}{4}\right) + f(\pi)\right]$$

$$\rightarrow \frac{\pi}{8}\left[0 + \sqrt{2} + 2 + \sqrt{2} + 0\right]$$

$$\rightarrow \frac{\pi}{8}\left[2 + 2\sqrt{2}\right] \approx 1.896$$



4.3 Riemann Sums

Jan 21

Warm up

(a) $\sum_{i=2}^{5} i^2 - 5 \to ((2^2-5)+(3^2-5)+(4^2-5)+(5^2-5))$

$\to -1+4+11+20 \to 14+20 \to 34$

(b) $\sum_{i=2}^{4} 3^i \to ((3^2)+3^3+3^4) \to 9+27+81$

$\to 81+36 \to 117$

(c) $\sum_{i=1}^{n} \frac{1}{n^3}(i^2+1) \to \frac{i^2+1}{n^3} \to \frac{1}{n^3}\sum_{i=1}^{n} i^2+1$

$\to \frac{1}{n^3}\left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 1\right) \to \frac{1}{n^3}\left(\left(\frac{n(n+1)(2n+1)}{6}\right)+\frac{n}{1}\right)$

$\to \frac{1}{n^2}\left(\frac{(n+1)(2n+1)}{6}+\frac{1}{1}\right) \to \frac{(n+1)(2n+1)+6}{6n^2} \to \frac{2n^2+3n+7}{6n^2}$

Ex 2: $\int_{-2}^{1} 2x\,dx$ (use upper sum): $\lim_{n\to\infty}(S(n))$

$\to f(x)=2x \quad \Delta x=\frac{3}{n} \quad M_i=-2+i\Delta x \to -2+\frac{3i}{n}$

$\to \sum_{i=1}^{n} f(M_i)\Delta x = \sum_{i=1}^{n} f\left(-2+\frac{3i}{n}\right)\frac{3}{n}$

$\to \frac{3}{n}\sum_{i=1}^{n}\left(2\left(-2+\frac{3i}{n}\right)\right) \to \frac{3}{n}\left(\sum_{i=1}^{n}(-4)+\frac{6}{n}\sum_{i=1}^{n} i\right)$

$\to \frac{3}{n}\left(-4n+\frac{6}{n}\left(\frac{n(n+1)}{2}\right)\right) \to \frac{3}{n}(-4n+3n+3)$

$\to \frac{-12n+9n+9}{n} \to \lim_{n\to\infty}\frac{-3n+9}{n} \to \boxed{-3}$

$\int_{-2}^{1} 2x\,dx \to [x^2]_{-2}^{1} \to (1-(-2)^2) \to 1-4 \to -3$

AP Question

t (hours)	0	1	3	6	8
$R(t)$ (liters/hr)	1340	1190	950	740	700

$W(t) = 2000e^{-t^2/20}$ on $0 \le t \le 8$ (t in hours)

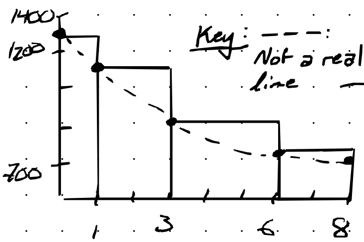
Pumped into tank (lt/hr); $R(t)$ = water removed, decreasing

@ $t=0$, 50,000 l in tank

ⓐ Estimate $R'(2)$.

$$R'(2) = \frac{R(3) - R(1)}{3-1} \rightarrow \boxed{-120 \; l/hr^2}$$

ⓑ left RM Sum, 4 partitions; is over- or underestimate?



See the decreasing function

$\boxed{\text{8050l, overestimate as R(t) is decreasing on 0 to 8, and a left sum on a decreasing function yields an overestimation}}$

$$\int_0^8 R(t)\,dt = (1-0)f(0) + (3-1)f(1) + (6-3)f(3) + (8-6)f(6)$$

$$= 1340 + 2(1190) + 3(950) + 2(740)$$

$$= 8050$$ (decreasing, overest.)

Chapter 4 Quiz Review

Points to study: ★ Integrating trig funcs.

↳ ★ Definite & indefinite

Yippee!

★ Integrals using "u" substitution

★ Findinding limits @ infinity of sums

★ Second fundamental theorem of Calc

★ Average & mean integrals

(10) $\int \frac{-\sec x \tan x}{\sqrt{\sec x}}\,dx \to u = \sec x \quad du = \sec x \tan x\,dx$

$-1\int \frac{1}{\sqrt{u}}\,du \to -1\int u^{-1/2}\,du \to -1\left[2u^{1/2}\right]+C \to \boxed{-2\sqrt{\sec x}+C}$

(12) $\int \frac{2x}{\sqrt{x+1}}\,dx \to \begin{matrix} u = x+1 \\ du = dx \end{matrix} \Rightarrow x = u-1$

$\to 2\int \frac{x}{\sqrt{x+1}}\,dx \to 2\int \frac{u-1}{\sqrt{u}}\,du \to 2\int \left(u^{1/2} - u^{-1/2}\right)du$

$\to 2\left(\frac{2u^{3/2}}{3} - 2u^{1/2}\right)+C \to \boxed{\frac{4(x+1)^{3/2}}{3} - 4(x+1)^{1/2}+C}$

(6) $s(n) = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\left(\frac{2}{n}\right)$ find lim. of $s(n)$ as $n\to\infty$

$\to s(n) = \sum_{i=1}^{n} f(m_i)\Delta x \to s(n) = \sum_{i=1}^{n}\left(1+\frac{i}{n}\right)^2\left(\frac{2}{n}\right)$

$\to \frac{2}{n}\sum_{i=1}^{n}\left(1+\frac{2i}{n}+\frac{i^2}{n^2}\right) \to \frac{2}{n}\left(\sum_{i=1}^{n}1 + \frac{2}{n}\sum_{i=1}^{n} i + \frac{1}{n^2}\sum_{i=1}^{n} i^2\right)$

$= \frac{2}{n}\left(n \cdot \frac{6n}{6n} + \frac{2}{n}\left(\frac{n(n+1)}{2}\right) + \frac{1}{n^{3}}\left(\frac{\cancel{n}(n+1)(2n+1)}{6}\right)\right)$

$\to \frac{\cancel{2}}{n}\left(\frac{6n^2+6n^2+6n+2n^2+3n+1}{3\cancel{6}n}\right)$

$= \frac{14n^2+9n+1}{3n^2} \to \lim_{n\to\infty} s(n) = \frac{14}{3}$

(4) $y' = 4x+4$ solve for $y=f(x)$

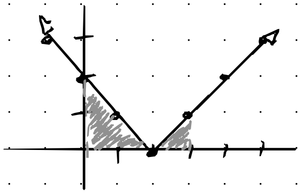
$$dx \cdot \frac{dy}{dx} = (4x+4)\,dx \to \int dy = \int (4x+4)\,dx$$

$$\to y = \frac{4x^2}{2} + 4x + C \to y = 2x^2+4x+C$$

$$\to 15 = 2(2^2)+4(2)+C$$

$$\to C = -1 \to \boxed{f(x)=2x^2+4x-1}$$

(11) $\int_0^3 |x-2|\,dx \to y=|x-2|$ vertex $=(2,0)$

→ 

$$\frac{1}{2}(2)(2)+\frac{1}{2}(1)(1) = 2+\frac{1}{2} = \frac{5}{2}$$

$$\int_0^2(-x+2)\,dx + \int_2^3 (x-2)\,dx$$

$$\to \left[\frac{-x^2}{2}+2x\right]_0^2 + \left[\frac{x^2}{2}-2x\right]_2^3$$

$$\to [2-0]+\left[\left(\frac{9}{2}-6\right)-(-2)\right] = \frac{5}{2}$$

(8) Average value of $f(x)=\cos x$ on $[0,\frac{\pi}{4}]$

$$\to \frac{1}{\frac{\pi}{4}-0}\int \cos x\,dx \to \frac{4}{\pi}\left[\sin\frac{\pi}{4} - \sin 0\right] \to \frac{4}{\pi}\left[\frac{\sqrt{2}}{2}+0\right]$$

$$\to \frac{2}{\pi}\left(\frac{\sqrt{2}}{1}\right) \to \boxed{\frac{2\sqrt{2}}{\pi}}$$

(3) $\int\left(\frac{x^2-x}{\sqrt{x}}\right)dx \to \frac{x^2}{x^{1/2}} - \frac{x}{x^{1/2}} \to x^{3/2}-x^{1/2}$

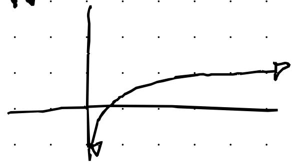
$$\int\left(x^{3/2}-x^{1/2}\right) \to \boxed{\frac{2x^{5/2}}{5} - \frac{2x^{3/2}}{3}+C}$$

(7) $\frac{d}{dx}\int_1^{2x^3}(t-3)^4\,dt \to \boxed{(2x^3-3)^4(6x^2)}$

FTC II

5.1 Natural Logs: Differentiation

$\ln x = (\log_e) x$



Properties of ln:

1. Domain is $(0, \infty)$ & range is $(-\infty, \infty)$
2. Function is increasing, continuous, & one to one
3. Concave downward

$$\boxed{\begin{array}{l} y = \log_b x \to b^y = x \\ n = \log_m u \to m^n = u \end{array}}$$

More properties:

1. $\ln 1 = 0$
2. $\ln(ab) = \ln a + \ln b \to$ because: $a^m \cdot a^n = a^{m+n}$
3. $\ln a^n = n \ln a$
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b \to$ because $\frac{a^m}{a^n} = a^{m-n}$

Ex 1:

a. $\ln\left(\frac{10}{9}\right) \to \ln(10) - \ln(9)$

b. $\ln\sqrt{3x+2} \to \ln(3x+2)^{1/2} \to \frac{1}{2}\ln(3x+2)$

c. $\ln \frac{6x}{5} \to \ln 6x - \ln 5 \to \ln 6 + \ln x - \ln 5$

d. $\ln \frac{(x^2+3)^2}{x(\sqrt[3]{x^2+1})} \to \ln(x^2+3)^2 - \ln\left(x(\sqrt[3]{x^2+1})\right)$

$\to 2\ln(x^2+3) - \left(\ln x + \frac{1}{3}\ln(x^2+1)\right)$

$\to 2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)$

Ex 2:

a. $\ln 2 = 0.693$

b. $\ln 32 = 3.466$

c. $\ln 0.1 = -2.303$

Definition of ln functions

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0$$

Definition of e

$$\ln e = \int_1^e \frac{1}{t}\,dt = 1$$

Derivative	Integral
$\frac{d}{dx}(\ln x) = \boxed{\frac{1}{x}}$	$\int \frac{1}{x}\,dx = \ln\lvert x\rvert + C$
$\frac{d}{dx}(\ln u) = \frac{1}{u}\cdot\frac{du}{dx} = \boxed{\frac{u'}{u}}$	$\int \frac{1}{u}\,du = \ln\lvert u\rvert + C$

Ex 3:

a. $\frac{d}{dx}[\ln(2x)] = \frac{2}{2x} \to \frac{1}{x}$

b. $\frac{d}{dx}[\ln(x^2+1)] = \frac{2x}{x^2+1}$

c. $\frac{d}{dx}[x\ln x] \to x\left(\frac{1}{x}\right) + 1\ln x \to \boxed{1 + \ln x}$

d. $\frac{d}{dx}[(\ln x)^3] \to 3(\ln x)^2\left(\frac{1}{x}\right) = \boxed{\frac{3(\ln x)^2}{x}}$ or $\frac{3}{x}(\ln x)^2$

Ex 4: find $f'(x)$

$f(x) = \ln\sqrt{x+1}$

$\to \frac{1}{2}\ln(x+1) \to f'(x) = \frac{1}{2} \cdot \frac{1}{(x+1)} = \boxed{\frac{1}{2(x+1)}}$

Ex 5:

$f(x) = \ln \frac{x(x^2+1)^2}{\sqrt{2x^3-1}} \to \ln(x(x^2+1)^2) - \frac{1}{2}\ln(2x^3-1)$

$\to f'(x) = \frac{1}{x} + 2\left(\frac{2x}{x^2+1}\right) - \frac{1}{2}\left(\frac{6x^2}{2x^3-1}\right)$

$\to \boxed{\frac{1}{x} + \frac{4x}{x^2+1} - \frac{3x^2}{2x^3-1}}$

5.2

Warm up: 1) $y = (3x)\ln(x^2-2)$

$\to (3x)\left(\frac{2x}{x^2-2}\right) + (3)(\ln(x^2-2))$

$\to \frac{6x^2}{x^2-2} + \ln(x^2-2)^3$

2) $g(x) = \ln\left(\frac{(1+x^2)^2}{\sqrt{4x-1}}\right) - \cancel{x}2\ln(1+x^2) - \left(\frac{1}{2}\ln(4x-1)\right)$

$\to 2\left(\frac{2x}{1+x^2}\right) - \frac{1}{2}\left(\frac{4}{4x-1}\right) \to \frac{4x}{1+x^2} - \frac{4}{8x-2}$

$\to \frac{4x}{1+x^2} - \frac{2}{4x-1} \to \underbrace{\frac{16x^2-4x}{4x-1+4x^3-x^2} - \frac{2+2x^2}{4x-1+4x^3-x^2}}$

$\to \frac{14x^2-4x-2}{4x^3-x^2+4x-1} = g'(x) \} \longrightarrow$ My special Brain

5.2 Natural Log Integration

Use log or ln differentiation when base & exponent are both a function of "x"

Eg.: $\frac{dy}{dx}\left(y = x^{x-1}\right) \rightarrow$

On next test

Steps:

1. Take ln of both sides — $y = x^{x-1}$, $\ln y = \ln x^{x-1}$
2. Use a property to simplify the right side — $\ln y = (x-1)\ln x$
3. Differentiate both sides with respect to "x" — $\frac{d}{dx}(\ln y) = \frac{d}{dx}\left((x-1)\ln x\right)$

$$y \cdot \frac{y'}{y} = \left[(x-1)\left(\frac{1}{x}\right) + (1)(\ln x)\right] y$$

4. Isolate y': Multiply both sides by y to get y' by itself — $y' = y\left(\frac{x-1}{x} + \ln x\right)$
5. Substitute y with given — $y' = x^{x-1}\left(\frac{x-1}{x} + \ln x\right)$

Theorem 5.5: Log Rule for Integration

Let u be a differentiable function of "x."

1. $\int \frac{1}{x}\,dx = \ln|x| + C$
2. $\int \frac{1}{u}\,du = \ln|u| + C$

} Always add abs() ↳ Just in case

Ex 1: $\int \frac{2}{x}\,dx \rightarrow 2\int \frac{1}{x}\,dx = 2(\ln|x|) + C$

Ex 2: $\int \frac{1}{4x-1}\,dx(4)$ $u = 4x-1$ $du = 4\,dx$

$\Rightarrow \frac{1}{4}\left(\int \frac{1}{u}\,du\right) \rightarrow \frac{1}{4}\ln|u| + C$

Ex 3: Find the area of $y=\frac{x}{x^2+1}$, ~~bounded~~ by x-axis & $x=3$

$\frac{1}{2}\int_0^3 \frac{x}{x^2+1}\,dx(2)$ $\quad u=x^2+1 \quad du=2x\,dx$ $\quad x=0,\ u=1;\ x=3,\ u=10$

$\to \frac{1}{2}\int_1^{10} \frac{1}{u}\,du \to \frac{1}{2}\left[\ln|u|\right]_1^{10}$

$\to \frac{1}{2}\left[\ln 10 - \ln 1\right] \to \boxed{\left(\frac{1}{2}\ln(10)\right) = \text{Area}}$

Ex 4: (a) $\int \frac{3x^2+1}{x^3+x}\,dx \to u=x^3+x \quad du=(3x^2+1)\,dx$

$\to \int \frac{1}{u}\,du \to \ln|u|+c \to \boxed{\ln|x^3+x|+c}$

(b) $\int \frac{\sec^2 x}{\tan x}\,dx \quad u=\tan x \quad du=\sec^2 x\,dx$

$\to \int \frac{1}{u}\,du \to \ln|u|+c \to \boxed{\ln|\tan x|+c}$

(c) $\frac{1}{2}\int \frac{x+1}{x^2+2x}\,dx\cdot 2 \quad u=x^2+2x \quad du=(2x+2)\,dx$

$\to \frac{1}{2}\int \frac{1}{u}\,du \to \frac{1}{2}\ln|u|+c \to \frac{1}{2}\ln|x^2+2x|+c$

(d) $\int \frac{1}{3x+2}\,dx \quad u=3x+2 \quad du=3\,dx$

$\to \frac{1}{3}\int \frac{1}{u}\,du \to \frac{1}{3}\ln|u|+c \to \frac{1}{3}\ln|3x+2|+c$

(e) $\int \frac{x^2+x+1}{x^2+1}\,dx$ long division

exponents are equal!

↓

DIVIDE

$\to x^2+1\overline{)\,x^2+x+1}\quad 1+\frac{x}{x^2+1} \to \int\left(1+\frac{x}{x^2+1}\right)dx$

$1(x^2+1)\quad \cancel{x^2}\quad 1$

$+x$ remainder

$\rightarrow \int 1\,dx + \int \frac{x}{x^2+1}\,dx \qquad u = x^2+1 \quad du = 2x\,dx$

$\rightarrow x + \frac{1}{2}\int \frac{1}{u}\,du \rightarrow x + \frac{1}{2}\ln|u| + C$

$\rightarrow \boxed{x + \frac{1}{2}\ln|x^2+1| + C}$

(f) $\int \frac{2x}{(x+1)^2}\,dx \qquad u = x+1 \quad du = 1\,dx$

$x = u-1$

$\rightarrow 2\int \frac{u-1}{u^2}\,du \rightarrow 2\int\left(\frac{1}{u} - \frac{1}{u^2}\right)du$

$\rightarrow 2\int\left(\frac{1}{u} - u^{-2}\right)du \rightarrow 2\left(\ln|u| - (-u^{-1})\right) + C$

$\rightarrow \boxed{2\left(\ln|x+1| + \frac{1}{u}\right) + C}$

(g) $\left(\frac{dy}{dx} = \frac{1}{x\ln x}\right) \cdot dx \rightarrow \int dy = \int \frac{1}{x\ln x}\,dx \qquad u = \ln x \quad du = \frac{1}{x}\,dx$

$\rightarrow y = \int \frac{1}{u}\,du \rightarrow y = \ln|u| + C$

$\rightarrow \boxed{y = \ln|\ln|x|| + C}$

(h) $\int \tan x\,dx \rightarrow (-1)\int \frac{\sin x\,(-1)}{\cos x}\,dx \qquad u = \cos x\,dx \quad du = -\sin\,dx$

$\rightarrow -\int \frac{1}{u}\,du \rightarrow -\ln|u| + C \rightarrow \boxed{-\ln|\cos x| + C}$

$\rightarrow \ln|\cos x|^{-1} + C \rightarrow \ln\left|\frac{1}{\cos x}\right| + C \rightarrow \boxed{\ln|\sec x| + C}$

Integrals of Trig. Funcs.

$\int \sin u \, du = -\cos u + c$ $\quad \int \cos u \, du = \sin u + c$

$\int \tan u \, du = -\ln|\cos u| + c$ $\quad \int \cot u \, du = \ln|\sin u| + c$

$\int \sec u \, du = \ln|\sec u + \tan u| + c$ $\quad \int \csc u \, du = -\ln|\csc u + \cot u| + c$

Ex 10: $\int_0^{\pi/4} \sqrt{1+\tan^2 x} \, dx$

$\to \int_0^{\pi/4} \sqrt{\sec^2 x} \, dx \to \left[\ln|\sec x + \tan x|\right]_0^{\pi/4}$

$\to \left[\left(\ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right|\right) - \left(\ln|\sec 0 + \tan 0|\right)\right]$

$\to \left[\left(\ln|\sqrt{2}+1|\right) - \left(\ln(1+0)\right)\right] = \boxed{\ln|\sqrt{2}+1|}$

---

on 43, remove x from "9" in denominator

5.3 Warm-up

a) $\int \frac{4x^2}{x^3-7} dx \quad u = x^3-7 \quad du = 3x^2 \, dx$

$\to \frac{4}{3} \int \frac{1}{u} du \to \boxed{\frac{4}{3} \ln|x^3-7| + c}$

b) $\frac{1}{5} \int \csc(5x) \, dx \to u = 5x \quad du = 5 \, dx$

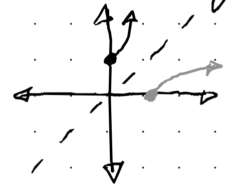
$\ln|\csc(5x) + \cot(5x)| + c$

$\boxed{-\frac{1}{5} \ln|\csc(5x) + \cot(5x)| + c}$

5.3 Inverse Functions

Facts to remember

- Switch x & y of function $\begin{cases} f(x) \Rightarrow (3,2) \\ f^{-1}(x) \Rightarrow (2,3) \end{cases}$
- Graph is therefore reflected over $y=x$ (the origin)



- To verify inverses:

and $\left[\begin{array}{l} f(f^{-1}(x)) = x \\ f^{-1}(f(x)) = x \end{array}\right.$ $\quad f \circ g$ $\quad g \circ f$ must do both

- How to find the inverse function

− Switch x & y − Solve for y

Ex 1: Prove inverses

$f(x) = 2x^3 - 1 \qquad f^{-1}(x) = g(x) = \sqrt[3]{\frac{x+1}{2}}$

$\Rightarrow 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \Rightarrow 2\left(\frac{x+1}{2}\right) - 1 \Rightarrow x \checkmark$

$\Rightarrow \sqrt[3]{\frac{(2x^3-1)+1}{2}} - 1 \Rightarrow \sqrt[3]{\frac{2x^3}{2}} - 1 \Rightarrow \sqrt[3]{x^3} \Rightarrow x \checkmark$

$\therefore$ f & g are inverses of each other

The existence of an Inverse function

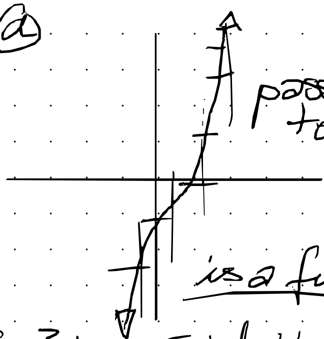
The inverse is only a function if:

ⓐ Must be "1 to 1", only 1 x per y & 1 y per x

ⓑ Strictly increasing / decreasing

Ex 2: $f(x) = x^3 + x - 1$ $\qquad$ $f(x) = x^3 - x + 1$

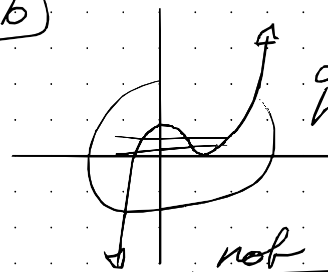
(a)



passes both tests

is a func.

(b)



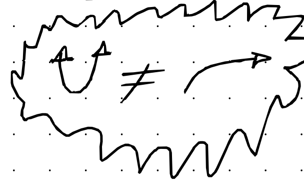
doesn't pass horizontal line test

not a func.

Ex 3: Find the inverse

$f(x) = \sqrt{2x-3} \to x = \sqrt{2y-3} \to x^2 = 2y - 3$

$\to x^2 + 3 = 2y \to y = = \frac{x^2+3}{2} =$



$\to f^{-1}(x) = \frac{1}{2}x^2 + \frac{3}{2}, \quad x \geq 0$

Properties of inverses

1. If "f" is continuous, "f^{-1}" is also continuous
2. If f is increasing, f^{-1} is also increasing
3. " decreasing, " decreasing
4. If f is diff'able on interval containing "c" & $f'(c) \neq 0$, then f^{-1} is diff'able at $f(c)$.

Derivative of inverse function

Let f be diff'able function, inverse (g) is diff'able at any x where $f'(g(x)) \neq 0$. Moreover

$$g'(x) = \frac{1}{f'(g(x))} \qquad f'(g(x)) \neq 0$$

Ex 5: $f(x) = \frac{1}{4}x^3 + x - 1$ ⓐ What is $f^{-1}(x)$ @ $x=3$

ⓑ What is $(f^{-1})'(x)$ @ $x=3$

(find slope of $f^{-1}(x)$ @ $x=3$)

Remember switched x & y!

$$(f^{-1})'(3) = \frac{1}{f'(f^{-1}(3))}$$

	x	y
f	? 2	3
f^{-1}	3	? 2

$$3 = \frac{1}{4}x^3 + x - 1 \rightarrow \frac{1}{4}x^3 + x - 4 = 0$$

(q = $\frac{1}{4}$, p = -4)

$$\frac{p}{q} = \frac{\pm 1, \pm 2, \pm 4}{\pm 1, \pm\frac{1}{2}, \pm\frac{1}{4}} = \pm 1, \pm 2, \pm 4$$

ⓐ $\boxed{\rightarrow 2}$

$$\begin{array}{c|cccc} 1 & \frac{1}{4} & 0 & 1 & -4 \\ & \downarrow & \frac{1}{4} & \frac{1}{4} & \\ \hline & \frac{1}{4} & \frac{1}{4} & & \end{array} \qquad \begin{array}{c|cccc} 2 & \frac{1}{4} & 0 & 1 & -4 \\ & \downarrow & \frac{1}{2} & 1 & 4 \\ \hline & \frac{1}{4} & \frac{1}{2} & 2 & 0 \end{array}$$

ⓑ $(f^{-1})'(3) = \frac{1}{f'(f^{-1}(x))} \rightarrow \frac{1}{f'(2)} \rightarrow \frac{1}{\frac{3}{4}(4)+1}$

$\rightarrow \boxed{\frac{1}{4}}$ slope of f^{-1} @ $x=3$

Ex 6: $f(x) = x^3$ for $(x \geq 0)$, $f^{-1}(x) = \sqrt{x}$

show that slopes of $f(x)$ & $f^{-1}(x)$ are reciprocals

@ ⓑ $(3,9)$ and $(9,3)$

$f(x) \rightarrow (3,9)$ $f'(3) =$

$f^{-1}(x) \rightarrow (9,3)$ $(f^{-1})'(9) =$

$(f^{-1})'(x) = \frac{1}{2\sqrt{x}}$

$(f^{-1})'(9) = \frac{1}{2\sqrt{9}} = \boxed{\frac{1}{6}}$

$\boxed{6 \text{ \& } \frac{1}{6} \text{ are reciprocals}}$

$f'(x) = 2x$

$f'(3) = 2(3) = 6$

$\rightarrow f(x) = x^2$ $y = \sqrt{x}$ $x \geq 0$

$x = y^2$

$f^{-1}(x) = \sqrt{x}$

$f^{-1}(9) = \boxed{3}$

5.4 Differentiation & Integration of Exponential Funcs.

Warm-up: $f(x) = 4x^3 + 3x - 4$, $(a = 3) \rightarrow$ slope of inverse at $x = 3$

$\rightarrow f'(x) = 12x^2 + 3$ $\quad f(1) = 3 \rightarrow 3 = 4x^3 + 3x - 4$

$\rightarrow 0 = 4x^3 + 3x - 7$

$\rightarrow (f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} \rightarrow f^{-1}(3) = 1$

$\rightarrow \dfrac{1}{f'(1)} \rightarrow \dfrac{1}{12(1)^2 + 3} \rightarrow \dfrac{1}{12+3} \rightarrow \boxed{\dfrac{1}{15}}$

	x	y
f	1	3
f^{-1}	3	1

$3 = 4x^3 + 3x - 4 \rightarrow 0 = 4x^3 + 3x - 7$ (q = 4, p = 7)

$\dfrac{p}{q} = \dfrac{\pm 1, \pm 7}{\pm 1, \pm 2, \pm 4} = \pm 1, \pm \frac{1}{2}, \pm \frac{1}{4}, \pm 7, \pm \frac{7}{2}, \pm \frac{7}{4}$ } Must use p's & q's

1 | 4 0 3 −7
 ↓ 4 4 7
4 4 7 0 $\rightarrow 4x^2 + 4x + 7$

$f(x) = e^x$ $\quad f'(x) = ?$ $\quad \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$h = .001$ $\quad \rightarrow \boxed{f'(x) = e^x}$ $\quad = \lim_{h \to 0} \dfrac{((e^{x+h}) - (e^x))}{h}$

$y_1 = e^x$

$y_2 = ((e^{x+.001}) - (e^x))/.001$

Inverse of "ln" func. is called natural exponential func., it's denoted by $f^{-1}(x) = e^x$. $y = e^x$ iff $x = \ln y$

Operations with exponential functions

1. $e^a \cdot e^b = e^{(a+b)}$
2. $e^a / e^b = e^{(a-b)}$

Ex1: $7 = e^{x+1} \rightarrow \ln 7 = \ln(e^{x+1})$

$\rightarrow \ln 7 = (x+1) \ln e \rightarrow \ln 7 = (x+1)(1) = x+1$

$\rightarrow (\ln 7) - 1 = x$

Ex 2: Solve

$\ln(2x-3) = 5 \rightarrow 2x-3 = e^5$

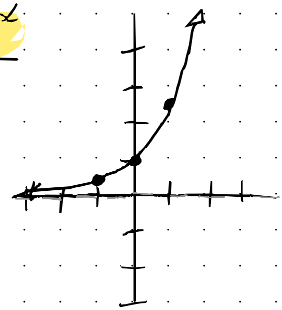
$\rightarrow x = \frac{(e^5+3)}{2}$

Sketching $f(x) = e^x$

x	$f(x)$
-1	e^{-1}
0	1
1	e^1

$\rightarrow e^{-1} \approx .4$

$\rightarrow e \approx 2.7$



Properties of e functions

1. domain $(-\infty, \infty)$, range $(0, \infty)$
2. Continuous, increasing, one-to-one
3. Concave up
4. $\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to -\infty} e^x = 0$

Derivative of e functions

1. $\frac{d}{dx}[e^x] = e^x$
2. $\frac{d}{dx}[e^u] = e^u \frac{du}{dx} = e^u \cdot u'$

Ex 3: ⓐ

$y = e^{2x-1} \rightarrow y' = e^{2x-1} \frac{du}{dx} \rightarrow e^{2x-1} \cdot 2 \rightarrow \boxed{2e^{2x-1}}$

ⓑ $y = e^{-\frac{3}{x}} \rightarrow u = -3x^{-1} \quad u' = \frac{3}{x^2}$

$\rightarrow e^{-3/x} \cdot \frac{3}{x^2} \rightarrow \boxed{\frac{3e^{-3/x}}{x^2}}$

Ex 4: $f(x) = xe^{x} \to \boxed{f'(x) = xe^{x} + 1e^{x}}$

$\to 0 = e^{x}(x+1) \to e^{x} \neq 0, \quad x+1 = 0 \to x = -1$

Testing $\to D: (-\infty, \infty) \to (-\infty, -1)(-1, \infty)$

$-2 \qquad 0$

$\to -2e^{-2} + e^{-2} \to \frac{-2}{e^{2}} + \frac{1}{e^{2}} \to (-)$

$\to 0e^{0} + 0^{e} \to 0 \to +$

Relative min. is @ $\frac{-1}{e}$ when $x = -1$.

$f(-1) = -1e^{-1}$

Integration of e funcs.

1. $\int e^{x}\,dx = e^{x} + C$
2. $\int e^{u}\,du = e^{u} + C$

Ex 4: $\int e^{3x+1}\,dx \to \quad u = 3x+1, \; du = 3dx$

$\frac{1}{3}\int e^{u}\,du \to \frac{1}{3}[e^{u}] + C \to \boxed{\frac{1}{3}e^{3x+1} + C}$

Ex 5:

$\int 5xe^{-x^{2}}\,dx \qquad u = -x^{2} \quad du = -2x\,dx$

$\frac{5}{-2}\int xe^{-x^{2}}\,dx\,{-2} \to -\frac{5}{2}\int e^{u}\,du \to -\frac{5}{2}(e^{-x^{2}}) + C$

Ex 6: ⓐ $\int \frac{e^{1/x}}{x^{2}}\,dx \to u = 1/x \quad du = \frac{-1}{x^{2}}\,dx$

$\to -\int e^{u}\,du \to \boxed{-e^{1/x} + C}$